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Instability due to heat diffusion in a stably stratified fluid

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A thermally conducting body floating in a fluid that is stably stratified by heat may develop unstable oscillations provided the body temperature shows a large enough time-lag relative to the fluid temperature. A gravitationally stable, non-Boussinesq fluid may itself become unstable in an oscillatory way through a similar diffusive time-lag. One case, that is investigated in the present work, occurs when the basic vertical temperature gradient $\beta = dT/dz$ and the thermal expansion coefficient α vary in an opposite sense: β is large when α is small and vice versa. The variation in β is here assumed to be caused by internal heat sources and sinks.

If vertical oscillations are started in such a fluid temperature variations will be produced in the regions of large β but the buoyancy forces do not develop until these perturbations have diffused to regions of large α. With an appropriate lag, the buoyancy forces may give a positive work and the oscillations can grow.

Two models are investigated. The first one is a non-viscous two-layer model with $\beta = 0$, $\alpha = \alpha_0$ in one layer and $\beta = \beta_0$, $\alpha = 0$ in the other layer. For this model analytical results are derived. The second model is more realistic, having continuous profiles $\beta(z)$ and $\alpha(z)$, viscosity and horizontal boundaries. The case is studied by a numerical technique, solving the equations directly in time.

A discussion of the numerical method is given in an appendix by K. Holmåker.

1. Background

For a fluid that is heated uniformly above absolute stability can be proved, assuming that the temperature gradient β and the thermal expansion coefficient α are constant and positive (z is the vertical co-ordinate, counted positive upward). This is a trivial case of the classical Rayleigh problem. One may ask what happens in the more general case where β and α are functions of z, but still positive at every point to keep the fluid gravitationally stable. Such a variable β may occur through the dependence of the thermal diffusivity on the variables of state, or through an effect of internal heat sources and sinks (the second case is considered in the present study). A variable α may also occur through the dependence on the variables of state, particularly the temperature.

Judging from the Rayleigh problem one may feel that such situations are also stable. However, the proof of stability cannot be carried through in the case where β and α are general, positive functions of z. One must therefore consider the possibility that some new form of instability, differing from ordinary Rayleigh convection, occurs at least for certain profiles $\beta(z)$ and $\alpha(z)$.

It should be noticed that the development of an instability in a gravitationally stable fluid in no way contradicts the basic laws of mechanics. Energy exists in the form of heat, and since there is also a temperature difference in the system a certain part of this heat can be transformed to mechanical energy in accordance with the second law of thermodynamics.

There exist already examples of instability in gravitationally stable fluids. One case is the 'double-diffusive instability' demonstrated by Stern (1960) and later investigators. Such an instability can be simply shown in a laboratory by pouring hot salt water over cold fresh water of higher density. Another case is the unstable oscillations occurring in a stably stratified stellar atmosphere (see e.g. Baker 1960). The instability—that is an effect of the dependence of opacity on the variables of state—plays an important role in the theory of pulsating stars.

Several years ago the author suggested the existence of a 'single diffusive instability'. It was at that time believed that the instability should be of a cellular type. Frictional coupling between cells in regions of large β and small α, and regions of large α and small β, would be an important part of the mechanism. Attempts to demonstrate such an instability in a mercury–water system were, however, unsuccessful. (Possibly the presence of some nasty surface effects at the mercury–water interface prevents the necessary viscous coupling.)

At this stage the author decided to try some numerical experiments for a continuously stratified fluid with suitable chosen functions $\beta(z)$ and $\alpha(z)$. Such experiments were carried out in the summer of 1969 using the computer SAAB D 21 in Gothenburg, and were successful as unstable motions could be found. A special study of the stability of the numerical method was carried out to ensure that the observed instability was a real effect in the assumed physical system.

The instability obtained in the numerical experiments turned out to be of a different type from that expected, being oscillatory rather than cellular. Frictional coupling played a minor role and unstable oscillations could actually be found in the limit of zero Prandtl number.

A two-layer model was further analysed analytically. This model has one layer with $\beta = 0$, $\alpha = \alpha_0$, and one layer with $\beta = \beta_0$, $\alpha = 0$. Although drastically simplified it is felt that it provides some insight into the nature of the instability.

This article starts with a discussion of a simple body oscillator. The mechanism of the unstable oscillations in a fluid is discussed in physical terms, then the two models are described in detail. An appendix on the numerical method completes the paper.

2. The body oscillator

Consider a fluid that is heated uniformly from above to give a linear temperature distribution $T_0 = \beta z$. In this fluid a small, thermally conducting body is placed at its natural density level. The body will be subjected to a buoyancy force

$$B = gV(\rho_1 \alpha_1 T_1 - \rho_0 \alpha_0 T_0), \tag{1}$$

where g is the acceleration due to gravity, V the volume of the body, T the temperature, ρ a standard density, and α the thermal expansion coefficient. Subscript 0 refers to the fluid, 1 to the body. It is assumed that $\rho_0\alpha_0 > \rho_1\alpha_1$ to make sure that the body has a gravitationally stable equilibrium.

The body is now forced into small vertical oscillations of constant amplitude. There is a natural frequency at which the body can oscillate, but one may obtain other frequencies by attaching to the body an appropriate ideal spring (figure 1). If the co-ordinate z denotes the position of the body one may write $z = z_0 \sin \omega t$, $T_0 = \beta z_0 \sin \omega t$. The body temperature T_1 will show an attenuation and a phase lag relative to T_0, and one may write $T_1 = a(\omega)\beta z_0 \sin[\omega t - \phi(\omega)]$, where a is an attenuation factor and ϕ a phase angle. The work done by the buoyance force over one cycle is

$$W = \int_0^{2\pi/\omega} B\frac{dz}{dt}dt = gV\rho_1\alpha_1 \int_0^{2\pi/\omega} T_1 \frac{dz}{dt}dt \qquad (2)$$

$$= -\pi g V \rho_1 \alpha_1 \beta z_0^2 a \sin\phi$$

(note that the T_0 term does not contribute). If ϕ lies in the range 0 to π, W is negative and the oscillations must die out if not sustained from outside. On the other hand, if ϕ lies in the range π to 2π, W is positive, and the oscillations may be

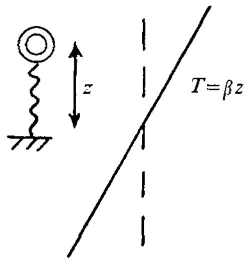

FIGURE 1. The body oscillator in an idealized case.

self-sustained even in the presence of friction. If one assumes a frictional force proportional to the velocity, $F = -r(dz/dt)$, the frictional work over one period is

$$D = \int_0^{2\pi/\omega} F\frac{dz}{dt}dt = -r\int_0^{2\pi/\omega}\left(\frac{dz}{dt}\right)^2 dt = -\pi\omega r z_0^2. \qquad (3)$$

The requirement $W + D > 0$ gives

$$\sin\phi < -\omega r / gV\rho_1\alpha_1\beta a. \qquad (4)$$

If $r < gV\rho_1\alpha\beta(a/\omega)$ there will exist an interval of phase angles between π and 2π that allow instability.

The above oscillator is a variant of the one discussed by Moore & Spiegel (1966). They consider a small body in a fluid that is *unstably* stratified by heat. In this case self-sustained oscillations may occur at small phase angles. If the body attains its maximum temperature after it passes its lower turning point and is rising, and its minimum temperature after it passes its upper turning-point and is falling, the buoyancy force will contribute positive work. It is, however, obvious that the same argument can be used for a fluid with reversed temperature stratification if the phase angle is increased by π.

Of course, there remains the practical question of how to obtain large enough phase angles. It can be shown that for a small, homogeneous sphere the phase lag for the volume averaged temperature (that, of course, is the meaning of the body temperature T_1) is always less than $\frac{1}{4}\pi$. Stability is therefore obtained when such a body hangs in a fluid heated from above. However, it is possible to surround the body with a shield of different material. If this material has a small thermal expansion coefficient one may obtain any wanted phase-lag by choosing an appropriate thickness of the shield.

The above oscillator is a simple example of a system producing mechanical energy by the action of heat diffusion. It should be noted that there is no potential energy available in the basic state, and that all mechanical energy therefore must come directly from the heat.

3. A possible fluid oscillator

Consider a fluid between two horizontal boundaries at a distance $2h$, the lower kept at temperature T_0 and the upper at temperature $T_0 + \Delta T$. It is assumed that the thermal expansion coefficient α varies, through its dependence on temperature or some other state variable. For example, let α be large in the upper portion of the fluid, small in the lower portion of the fluid. The vertical temperature gradient β is modified by introducing internal heat sources and sinks, with a vertical distribution that creates a small β in the upper portion and a large β in the lower portion of the fluid.

Consider now a standing gravity wave within this fluid, with a wavelength L (assumed to be larger than h). The period of the oscillation is of order

$$L[g(\Delta\rho/\rho)\,h]^{-\frac{1}{2}},$$

where $\Delta\rho/\rho$ is the characteristic relative density variation across the fluid. Focusing attention on a part of the oscillation where the motion is upward it is seen that a negative temperature anomaly will be produced in the upper portion of the fluid, according to the equation $\partial T/\partial t = -\beta W + \ldots$. However, this anomaly will not produce any local buoyancy force, since α is small. After some time, the anomaly has diffused into the lower portion, where, because of its low β, no thermal perturbations are directly produced. When the negative temperature anomaly reaches the lower portion a large negative buoyancy is set up. If this happens when the oscillation has turned and the motion is downward a positive work is produced. A similar argument applies for the other phase of the motion.

A requirement for unstable oscillations is obviously that the diffusive time h^2/κ (κ is the heat diffusivity) is comparable to the above oscillation time; that gives $g(\Delta\rho/\rho)\,h^3/\kappa^2 \sim (L/h)^2$. The parameter on the left-hand side is the overall Schmidt number S for the fluid. Thus, for a given S one may have unstable oscillations for scales of order $hS^{\frac{1}{2}}$. For sufficiently small S, friction must, however, become important. Below a certain critical Schmidt number all scales must be stable.

The instability suggested here is similar to the one of the body oscillator. In both cases the temperature perturbations are produced in one portion of the system and diffused with a certain delay to another portion where the buoyancy forces develop. The delay is in both cases chosen such that the buoyancy force is positive during upward motion, negative during the downward motion, producing net mechanical energy.

It may be noted that a similar argument can be developed for a travelling gravity wave. In this case one considers diffusive delays in space rather than in time. For example, looking on a crest of the travelling wave, one finds that a negative temperature anomaly is created in the upper portion of its front side. This diffuses down to produce a negative buoyancy force on the back of the wave, driving this forward. Similarly the downward motion on the back side creates a delayed upward buoyancy helping to rise the following crest.

4. The stability problem for a two-layer model

The simplest fluid model for which instability by diffusion has been proved is an unbounded two-layer model with $\beta = 0$, $\alpha = \alpha_0$ in one layer (say, the upper one), and $\beta = \beta_0$, $\alpha = 0$ in the other layer (figure 2).

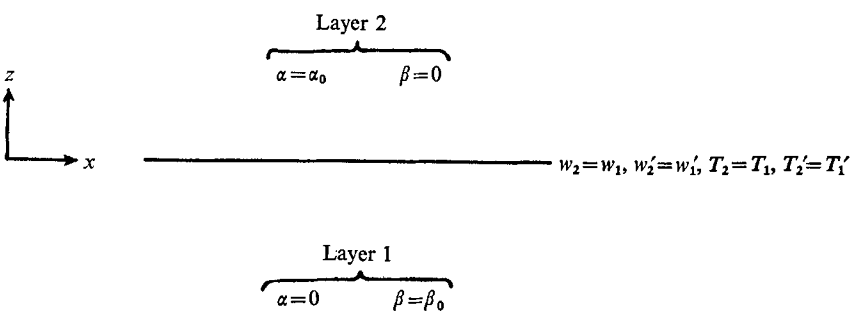

FIGURE 2. The two-layer model.

The Prandtl number is zero (zero viscosity and a constant thermal diffusivity) and the Boussinesq approximation is applied. The variation in β is assumed to be produced by an internal heat source at the interface.

The governing perturbation equations become

$$\frac{\partial \mathbf{v}_1}{\partial t} = -\frac{1}{\rho_0}\nabla p_1, \quad \nabla . \mathbf{v}_1 = 0, \quad \frac{\partial T_1}{\partial t} = -\beta_0 w_1 + \kappa \nabla^2 T_1, \qquad (5), (6), (7)$$

$$\frac{\partial \mathbf{v}_2}{\partial t} = -\frac{1}{\rho_0}\nabla p_2 + g\alpha_0 T_2, \quad \nabla . \mathbf{v}_2 = 0, \quad \frac{\partial T_2}{\partial t} = \kappa \nabla^2 T_2, \qquad (8), (9), (10)$$

where $\mathbf{v} = (u, v, w)$ is the velocity and g the acceleration due to gravity (in the negative z direction).

Subscript 1 denotes the lower layer, subscript 2 the upper layer. It is required that velocity and temperature vanish at large distances from the interface. At the interface one must match pressure, normal velocity, temperature and normal heat flux.

It has been assumed that no density jump occurs at the interface (the pressure is proportional to $\rho(dw/dz)$, and with a jump in density a jump in dw/dz is also required, in a normal mode form).

The thermal diffusion will, however, produce density perturbations in the upper layer. The model is, of course, unrealistic by the absence of a basic stable density gradient. However, the aim has been to simplify this model to the maximum degree. The stable density gradient, the viscosity and the horizontal boundaries will be included in a second model treated in the following section.

In the linearized problem the following conditions are applied at the undisturbed interface, $z = 0$:

$$p_2 = p_1, \quad w_2 = w_1, \quad T_2 = T_1, \quad \partial T_2/\partial z = \partial T_1/\partial z \quad (\text{at } z = 0). \qquad (11), (12), (13), (14)$$

The variables p, u, v are eliminated between the equations of motion and the continuity equation and the form of normal modes is introduced

$$w = W(z)\exp\left[i(k_x x + k_y y) + \sigma t\right], \qquad (15)$$

$$T = \Theta(z)\exp\left[i(k_x x + k_y y) + \sigma t\right], \qquad (16)$$

where k_x, k_y are horizontal wave-numbers and σ a complex frequency. The following equations and interface conditions follow for W and Θ:

$$\sigma\left(\frac{d^2}{dz^2} - k^2\right)W_1 = 0, \quad \sigma\Theta_1 = -\beta_0 W_1 + \kappa\left(\frac{d^2}{dz^2} - k^2\right)\Theta_1, \qquad (17), (18)$$

$$\sigma\left(\frac{d^2}{dz^2} - k^2\right)W_2 = -g\alpha_0 k^2 T_2, \quad \sigma\Theta_2 = +\kappa\left(\frac{d^2}{dz^2} - k^2\right)\Theta_2, \qquad (19), (20)$$

$$W_2 = W_1, \quad \frac{dW_2}{dz} = \frac{dW_1}{dz}, \quad T_2 = T_1, \quad \frac{dT_2}{dz} = \frac{dT_1}{dz}, \qquad (21), (22), (23), (24)$$

where $k = (k_x^2 + k_y^2)^{\frac{1}{2}}$ is a total horizontal wave-number.

From the above equations (17)–(20) one derives the following solutions, decaying at large distance from the interface:

$$W_1 = A\,e^{kz}, \quad \Theta_1 = -(\beta_0/\sigma)\,A\,e^{kz} + B\,e^{\lambda z}, \qquad (25), (26)$$

$$W_2 = (-g\alpha_0 k^2\kappa/\sigma^2)\,C\,e^{-\lambda z} + D\,e^{-kz}, \quad \Theta_2 = C\,e^{-\lambda z}, \qquad (27), (28)$$

where A, B, C, D are arbitrary constants and $\lambda = (k^2 + (\sigma/\kappa))^{\frac{1}{2}}$, the root with positive real part being chosen.

Applying the interface conditions (21)–(24) gives four homogeneous equations for A, B, C, D and a characteristic equation

$$\frac{1}{4}\frac{g\alpha_0\beta_0 k^2\kappa}{\sigma^3}\left(\frac{k}{\lambda} - 1\right)\left(\frac{\lambda}{k} - 1\right) = 1. \qquad (29)$$

Inserting back $\sigma = \kappa(\lambda^2 - k^2)$ and introducing $\lambda^* = \lambda/k$, $S_k = g\alpha_0\beta_0/\kappa^2 k^4$, the equation takes the form

$$\lambda^*(\lambda^{*2} - 1)(\lambda^* + 1)^2 + \tfrac{1}{4}S_k = 0. \qquad (29a)$$

Here a non-vanishing factor $(\lambda^* - 1)^2$ has been divided out.

The above equation determines five roots λ_i^* as function of S_k, which is a Schmidt number based on the scale k^{-1}. Instability occurs when at least one value λ_i^{*2} has a real part larger than 1, since this gives a positive real part for σ. To begin with it can be verified that small S gives stability, large S instability. For small S the roots to (29a) approach the values

$$\lambda_1^* = \tfrac{1}{4}S_k, \quad \lambda_2^* = 1-\tfrac{1}{32}S_k, \quad \lambda_{3,4,5}^* = -1-\tfrac{1}{2}S_R^{\frac{1}{3}}e^{i\frac{2}{3}n\pi} \quad (n = 0, 1, 2),$$

and all λ_i^{*2} have real parts smaller than 1. For large S the roots approach the values

$$\lambda_{1,2,3,4,5}^* = -(\tfrac{1}{4}S_k)^{\frac{1}{5}}e^{i\frac{2}{5}n\pi} \quad (n = 0, 1, 2, 3, 4).$$

Two of these roots ($n = 2, 3$) give instability. The corresponding real part of σ, representing the growth-rate, is $\kappa k^2(\frac{1}{4}S_k)^{\frac{2}{5}}\cos\frac{4}{5}\pi$ and the imaginary part of σ, representing the angular frequency of the oscillation, is $\pm\kappa k^2(\frac{1}{4}S_k)^{\frac{2}{5}}\sin\frac{4}{5}\pi$.

The marginal case, where $\mathrm{Re}(\lambda^{*2}) = 1$, must be represented by a neutral oscillation, as (29a) cannot be satisfied by the real root $\lambda^* = 1$. The corresponding value of S_k can be determined as follows. Put $\lambda^* = \alpha + i\beta$. In the marginal case one must have $\alpha^2 - \beta^2 = 1$. The requirement that the imaginary part of the equation (29a) should vanish gives $\alpha - \beta^2 = 0$, and one finds

$$\alpha = [\tfrac{1}{2}(1+\sqrt{5})]^2, \; \beta = \tfrac{1}{2}(1+\sqrt{5}).$$

Inserting this value in the real part of (29a) gives the critical value of S_k:

$$S_{kc} = 16\alpha\beta^2(1+\alpha)^2 = 720+320\sqrt{5} \simeq 1435{\cdot}5. \tag{30}$$

The Schmidt number based on the wavelength differs from the above by a factor $(2\pi)^4 \simeq 1558$, and takes on a critical value of about $2{\cdot}2\times10^6$.

The angular frequency of the neutral oscillation becomes

$$\omega_c = 2\alpha\beta\kappa k^2 = (\tfrac{19}{2}+\tfrac{17}{4}\sqrt{5})\kappa k^2 \simeq 19{\cdot}03\kappa k^2. \tag{31}$$

A generalization of the above problem is obtained by introducing horizontal boundaries. For instance, if boundaries where w and T vanish are introduced at $z = \pm h$, the solution is built up by terms of the form $\sinh k(z\pm h)$, $\cosh k(z\pm h)$. The characteristic equation obtained by applying the interface conditions becomes

$$\frac{4(\lambda^{*2}-1)^3}{S_k} = 2-\frac{\lambda^*\tanh a}{\tanh(a\lambda^*)}-\frac{\tanh(a\lambda^*)}{\lambda^*\tanh a}, \tag{32}$$

where $a = kh$. Again it can be verified that the marginal case must be a neutral oscillation (the right-hand side of the equation, which is of the form $2-x-(1/x)$, is negative for real positive arguments). In this problem one may introduce the Schmidt number $S = g\alpha_0\beta_0k^4/\kappa^2$, based on the distance h, and ask for the range of a values giving instability at a given S. For large S one can always find a range of large a values giving instability. This is the limiting case studied previously (for large a equation (32) reduces to (29a)). For small enough S values the problem is, on the other hand, stable for all a values. Therefore a certain critical S value should exist. The numerical determination of this value from (32) has not yet been made.

5. A numerical demonstration of the instability

It should be of interest to have the diffusive instability demonstrated for at least one realistic case, where β and α vary continuously over the depth, and where the effects of viscosity and horizontal boundaries are included. This has been achieved by some numerical experiments carried out on the SAAB D 21 computer at the University of Gothenburg, integrating the equations directly in time.

The functions β and α that are used are shown in figure 3 (a). In this case there exists a stable density stratification throughout the fluid. The coefficient of viscosity and the thermal diffusivity are assumed constant, and a partial Boussinesq approximation is used. As before, it is assumed that the variation in β is produced by an internal heat source, which in the present case is distributed over the depth. The model is described by the perturbation equations

$$\partial \mathbf{v}/\partial t = -(1/\rho_0)\nabla p + \mathbf{g}\alpha(z)\,T + \nu\nabla^2\mathbf{v}, \tag{33}$$

$$\nabla . \mathbf{v} = 0, \tag{34}$$

$$\partial T/\partial t = -\beta(z)\,w + \kappa\nabla^2 T, \tag{35}$$

where, as before, $\mathbf{v} = (u, v, w)$ is the velocity and $\mathbf{g}$ the gravity, in the negative z direction.

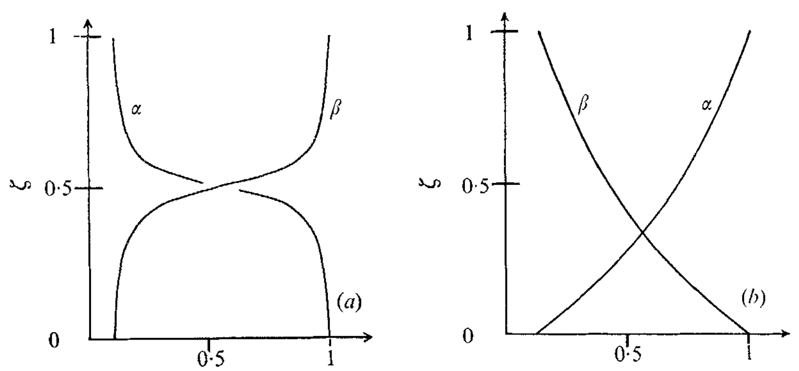

FIGURE 3. Vertical distribution of β and α (normalized). The idealized case used for the numerical experiment is shown in (a), a possible case for a water layer near 4 °C that is heated radiatively is shown in (b). Note that the idealized profiles may be reflected through the line $\zeta = 0{\cdot}5$.

Boundaries are introduced at $z = 0$ and D. These are assumed to be free (no tangential stress) and kept at constant temperatures. Eliminating u, v and p between (33), (34) one finds

$$\frac{\partial}{\partial t}(\nabla^2 w) = g\alpha(z)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)T + \nu\nabla^4 w. \tag{36}$$

Equations (35) and (36) determine w and T. The associated boundary conditions are

$$w = 0, \quad \partial^2 w/\partial z^2 = 0, \quad T = 0 \quad \text{at} \quad z = 0, D. \qquad (37), (38), (39)$$

Write

$$w = W(z,t)\exp\left[i(k_x x + k_y y)\right], \quad T = \Theta(z,t)\exp\left[i(k_x x + k_y y)\right] \quad (40), (41)$$

and define new variables for vertical distance, time and vertical velocity

$$\zeta = \frac{2\pi z}{D}, \quad \tau = \frac{4\pi^2\kappa t}{D^2}, \quad \omega = \frac{1}{a}\left(\frac{\beta_0}{g\alpha_0}\right)^{\frac{1}{2}} W,$$

where $a = kD/2\pi$ and α_0, β_0 are amplitudes of the functions α and β:

$$\alpha = \alpha_0\alpha^*(\zeta), \quad \beta = \beta_0\beta^*(\zeta).$$

Further let $S = g\alpha_0\beta_0 D^4/\kappa^2$ be an overall Schmidt number and $P = \nu/\kappa$ the Prantdl number.

The equations and boundary conditions then take the form

$$\frac{\partial}{\partial\tau}\left(\frac{\partial^2}{\partial\zeta^2}-a^2\right)\omega = -A\alpha^*(\zeta)\,\Theta + P\left(\frac{\partial^2}{\partial\zeta^2}-a^2\right)^2\omega, \tag{42}$$

$$\frac{\partial\Theta}{\partial\tau} = -A\beta^*(\zeta)\,\omega + \left(\frac{\partial^2}{\partial\zeta^2}-a^2\right)\Theta, \tag{43}$$

$$\omega = 0, \quad \partial^2\omega/\partial\zeta^2 = 0, \quad \Theta = 0 \quad \text{at} \quad \zeta = 0, 2\pi, \qquad (44), (45), (46)$$

where $A = (a/4\pi^2)\,S^{\frac{1}{2}}$.

These equations were integrated numerically in time, starting from the initial condition $\omega = \sin\zeta$, $\Theta = 0$.† The functions $\alpha^*(\zeta)$ and $\beta^*(\zeta)$ were represented by the expressions

$$\left.\begin{aligned} \alpha^* &= \tfrac{1}{2}+\delta+(\tfrac{1}{2}-\delta)\frac{\arctan m(\zeta-\pi)}{\arctan m\pi},\\ \beta^* &= \tfrac{1}{2}+\delta-(\tfrac{1}{2}-\delta)\frac{\arctan m(\zeta-\pi)}{\arctan m\pi} \quad \text{with} \quad \delta = 0{\cdot}05,\ m = 3. \end{aligned}\right\} \qquad (47, 48)$$

The parameters varied in the runs were P, S and a. The numerical scheme for the computations and a discussion of numerical stability and convergence are given in the appendix by Kjell Holmåker. In most of the computations the step-length was $\Delta\zeta = 2\pi/40$ (40 steps across the layer), $\Delta\tau = 0{\cdot}01$. The investigation given by Holmåker shows that $\Delta\tau/(\Delta\zeta)^2$ must be chosen smaller than $\frac{1}{2}$ and $1/2P$ to ensure numerical stability. To test the numerical accuracy certain cases were re-run with smaller steps both in time and space. From these tests it was concluded that the numerical error in the amplitude of the oscillations was of order 1 % after an integration time of 20 units (which typically gives 10 oscillations). The accuracy was considered sufficient for the purpose of demonstrating the physical instability. (If one wants to determine a critical curve by this type of experiment smaller steps should be used.)

The computations were made for three different Prandtl numbers, $P = 1$, $0{\cdot}1$ and $0{\cdot}01$. For sufficiently large Schmidt number instability was always obtained for the values of P and a attempted. The critical S value was in the range 10^6–10^7. As an example, the runs for the case $P = 0{\cdot}01$ are shown in figure 4.

Examples of curves for the kinetic energy are given in figure 5 (*a*)–(*c*), for one damped, one weakly unstable and one strongly unstable case. The weakly unstable case, in fact, was the first example of instability found in the numerical experiments. The variation of the vertical velocity and temperature perturbation over one cycle is shown in figure 6 for this case. The phase-lag between the temperature in the upper and the lower half of the fluid can be seen. In the first

† This initial state was chosen in an attempt to find certain unstable cellular motions. The oscillations found in the numerical experiment turned out to be badly represented by this state. They could still be detected, but for some weakly unstable modes the integration had to be carried over quite a long time.

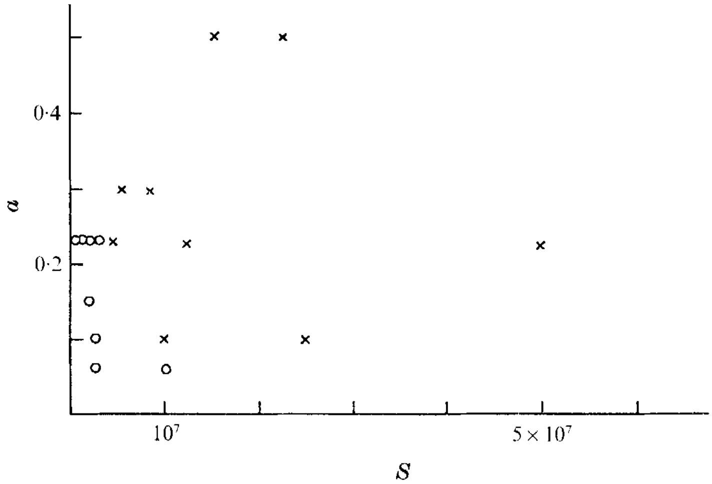

FIGURE 4. Cases run numerically for $P = 0{\cdot}01$. ○, stable case; ×, unstable case.

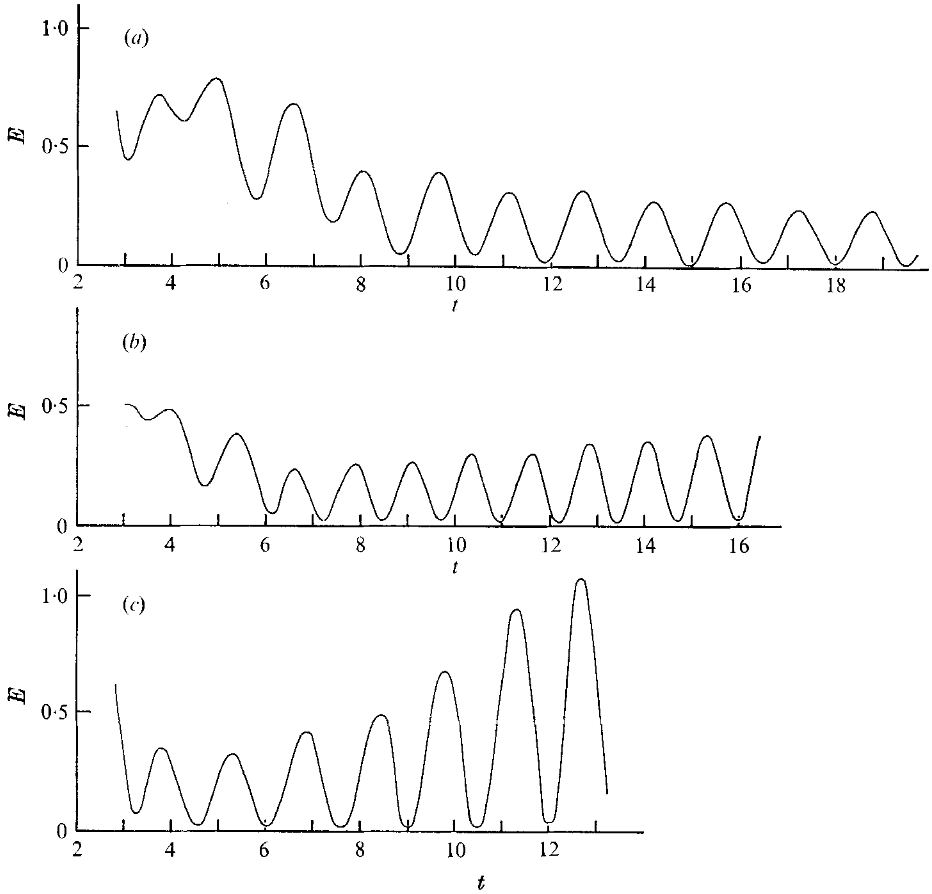

FIGURE 5. Kinetic energy as a function of time for $P = 0{\cdot}01$. (a) $S = 1{\cdot}12\times 10^6$, $a = 0{\cdot}224$; (b) $S = 10^7$, $a = 0{\cdot}1$; (c) $S = 7{\cdot}8\times 10^7$, $a = 0{\cdot}224$.

profiles one sees the negative temperature produced in the upper layer by upward motion, and how the perturbation diffuses into the lower layer. When the velocity has changed sign and the temperature in the upper layer is positive the lower-layer temperature is still negative (see $t = 10{\cdot}2$ and $10{\cdot}4$), contributing positively to the buoyancy work.

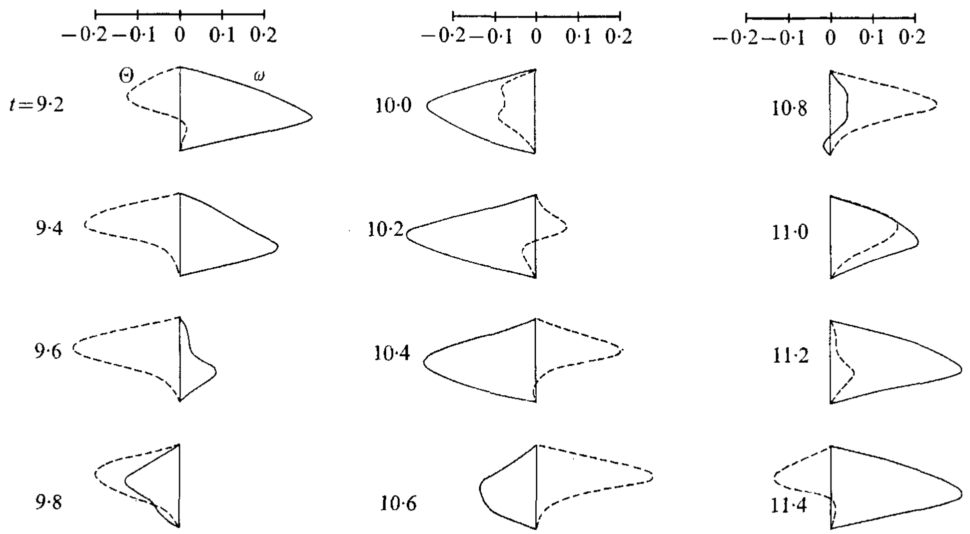

FIGURE 6. Time sequence over one cycle in the case $P = 0{\cdot}01$, $S = 10^7$, $a = 0{\cdot}1$ (case (*b*) in figure 5). ——, vertical velocity; – – –, perturbation temperature.

6. Possible applications

The diffusive instability, demonstrated here for some simplified models, has not yet been observed in laboratory experiments or in nature, to the author's knowledge. Certainly it requires very special conditions to develop. The profiles for β and α must vary in the right way, and the Schmidt number must be very large. The unwillingness of the fluid to become unstable the 'wrong way' is also reflected by the relatively weak growth-rates which are found.

Under laboratory conditions the instability may possibly be found for water; that is known to have a large variation in α near 4 °C. For a water layer kept close to 4 °C and heated radiatively such theoretical profiles for β and α as are shown in figure 3 (*b*) can be produced. The density stratification is stable everywhere. In geophysical situations there are possibilities of oscillations of a similar type. The author has earlier speculated about such oscillations in a coupled atmosphere-ocean model. In this case the difference in thermal expansion coefficient and thermal capacity of air and water plays a main role. The delay depends both on diffusive and radiative processes at the interface. It is, however, difficult to test this idea because of the complexity of the system, that includes turbulent motions of different scales, but the present model study may encourage further work on the problem.

The study of unstable oscillations in connexion with stellar problems was mentioned in § 1. In this case radiative effects play the main role. The delay is mainly local. If variations in β and α of the right type can be found in astrophysical situations it seems possible that oscillations of the type studied in this article can occur. Radiation rather than molecular diffusion would propagate the thermal anomalies from the generation region to the buoyancy-producing region.

Finally, one may think of cases where buoyancy forces are produced by other processes than heat. For example, one may look on situations where buoyancy is produced by material diffusion, by condensation or by chemical reactions. In all cases where a suitable delay occurs between the generation of the anomaly and the nearby production of the buoyancy force the possibility for an instability of the described type also exists.

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Appendix. Description of the numerical method with discussion of the convergence

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In this appendix a numerical solution of the equations (42)–(46) above is discussed. Thus, consider the following equations

$$\frac{\partial T}{\partial t} = \left(\frac{\partial^2}{\partial z^2} - a^2\right) T - g(z)\, w, \tag{A 1}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial z^2} - a^2\right) w = p\left(\frac{\partial^2}{\partial z^2} - a^2\right)^2 w - f(z)\, T, \tag{A 2}$$

$$w = \partial^2 w/dz^2 = T = 0 \quad \text{for} \quad z = 0 \quad \text{and} \quad z = 2\pi, \tag{A 3}$$

$$T = 0, \quad w = w_0(z) \quad \text{for} \quad t = 0. \tag{A 4}$$

It is known that there is a constant A such that $0 < f(z) \leqslant A$, $0 < g(z) \leqslant A$.

We seek an approximate solution of this system for $0 \leqslant t \leqslant t_0$. Let J be a positive integer and k a small positive number. Consider the net-points

$$(z, t) = (jh, nk), \quad \text{where} \quad h = 2\pi/J, \quad j = 0, 1, \ldots, J, \quad t = nk,$$

and n is an integer such that $0 \leqslant nk \leqslant t_0$. Introduce a new variable

$$\phi = \partial^2 w/\partial z^2 - a^2 w.$$

The resulting system of differential equations is then replaced by the following system of difference equations, namely

$$\frac{T_j^{n+1} - T_j^n}{k} = \frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{h^2} - a^2 T_j^n - g_j w_j^n \quad (j = 1, \ldots, J-1), \tag{A 5}$$

$$T_0^{n+1} = T_J^{n+1} = 0, \tag{A 6}$$

$$\frac{\phi_j^{n+1} - \phi_j^n}{k} = p\left(\frac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{h^2} - a^2\phi_j^n\right) - f_j T_j^{n+1} \quad (j = 1, \ldots, J-1), \tag{A 7}$$

$$\phi_0^{n+1} = \phi_J^{n+1} = 0, \tag{A 8}$$

$$\frac{w_{j+1}^{n+1} - 2w_j^{n+1} + w_{j-1}^{n+1}}{h^2} - a^2 w_j^{n+1} = \phi_j^{n+1} \quad (j = 1, \ldots, J-1), \tag{A 9}$$

$$w_0^{n+1} = w_J^{n+1} = 0, \tag{A 10}$$

$$T_j^0 = 0, \quad w_j^0 = w_0(jh), \quad \phi_j^0 = w_0''(jh) - a^2 w_0(jh) \quad (j = 0, \ldots, J). \tag{A 11}$$

Here T_j^n is supposed to approximate $T(jh, nk)$, etc., and f_j and g_j stand for $f(jh)$ and $g(jh)$.

With $\lambda = k/2$, where λ is constant when we let $h, k \to 0$, the equations (A 5), (A 7) and (A 9) become

$$T_j^{n+1} = \lambda T_{j+1}^n + (1 - 2\lambda - a^2k)\, T_j^n + \lambda T_{j-1}^n - kg_j w_j^n, \tag{A 12}$$

$$\phi_j^{n+1} = p\lambda\phi_{j+1}^n + (1 - 2p\lambda - pa^2k)\, \phi_j^n + p\lambda\phi_{j-1}^n - kf_j T_j^{n+1}, \tag{A 13}$$

$$w_{j+1}^{n+1} - (2 + a^2h^2)\, w_j^{n+1} + w_{j-1}^{n+1} = h^2\phi_j^{n+1}. \tag{A 14}$$

Let $\tilde{T}$, $\tilde{w}$ and $\tilde{\phi}$ be the exact solution of (A 1)–(A 4) and assume that these functions are twice continuously differentiable with respect to t and four times with respect to z for $0 \leqslant z \leqslant 2\pi$, $t \geqslant 0$. Then Taylor series expansions yield (with the notation $\tilde{T}_j^n = \tilde{T}(jh, nk)$, etc.)

$$\tilde{T}_j^{n+1} = \lambda\tilde{T}_{j+1}^n + (1 - 2\lambda - a^2k)\, \tilde{T}_j^n + \lambda\tilde{T}_{j-1}^n - kg_j\tilde{w}_j^n + O(h^4), \tag{A 15}$$

$$\tilde{\phi}_j^{n+1} = p\lambda\tilde{\phi}_{j+1}^n + (1 - 2p\lambda - pa^2k)\, \tilde{\phi}_j^n + p\lambda\tilde{\phi}_{j-1}^n - kf_j\tilde{T}_j^{n+1} + O(h^4), \tag{A 16}$$

$$\tilde{w}_{j-1}^{n+1} - (2 + a^2h^2)\, \tilde{w}_j^{n+1} + \tilde{w}_{j+1}^{n+1} = h^2\tilde{\phi}_j^{n+1} + O(h^4). \tag{A 17}$$

Now the purpose is to prove that the solutions of the difference equations converge to the solutions of the differential equations. To be more precise, consider a sequence of nets with mesh sizes h_ν, k_ν; $k_\nu/h_\nu^2 = \lambda$, $h_\nu \to 0$ as $\nu \to \infty$. Let (z, t) be a point belonging to all nets from a certain index ν_0. If $z = j_\nu h_\nu$, $t = n_\nu k_\nu$ for $\nu \geqslant \nu_0$, then the assertion is that $|\tilde{T}_{j_\nu}^{n_\nu} - T_{j_\nu}^{n_\nu}| \to 0$ as $\nu \to \infty$, etc.

Set $\qquad \epsilon_j^n = \tilde{T}_j^n - T_j^n, \quad \delta_j^n = \tilde{\phi}_j^n - \phi_j^n, \quad \eta_j^n = \tilde{w}_j^n - w_j^n$

and subtract (A 12), (A 13), (A 14) from (A 15), (A 16), (A 17). Then one obtains

$$\epsilon_j^{n+1} = \lambda\epsilon_{j+1}^n + (1 - 2\lambda - a^2k)\, \epsilon_j^n + \lambda\epsilon_{j-1}^n - kg_j\eta_j^n + O(h^4), \tag{A 18}$$

$$\epsilon_0^{n+1} = \epsilon_J^{n+1} = 0, \tag{A 19}$$

$$\delta_j^{n+1} = p\lambda\delta_{j+1}^n + (1 - 2p\lambda - pa^2k)\, \delta_j^n + p\lambda\delta_{j-1}^n - kf_j\epsilon_j^{n+1} + O(h^4), \tag{A 20}$$

$$\delta_0^{n+1} = \delta_J^{n+1} = 0, \tag{A 21}$$

$$\eta_{j-1}^{n+1} - (2 + a^2h^2)\, \eta_j^{n+1} + \eta_{j+1}^{n+1} = h^2\delta_j^{n+1} + O(h^4), \tag{A 22}$$

$$\eta_0^{n+1} = \eta_J^{n+1} = 0, \tag{A 23}$$

$$\epsilon_j^0 = 0, \quad \delta_j^0 = 0, \quad \eta_j^0 = 0 \quad (j = 0, \ldots, J). \tag{A 24}$$

In (A 18), (A 20) and (A 22) j runs from 1 to $J-1$.

Set $\|\epsilon^n\| = \max_{0\leqslant j\leqslant J}|\epsilon_j^n|$, etc., and suppose $\lambda \leqslant \min(\frac{1}{2}, 1/2p)$. Then

$$\|\epsilon^{n+1}\| \leqslant (1+a^2k)\|\epsilon^n\| + kA\|\eta^n\| + O(h^4), \qquad \text{(A 25)}$$

$$\|\delta^{n+1}\| \leqslant (1+pa^2k)\|\delta^n\| + kA\|\epsilon^{n+1}\| + O(h^4). \qquad \text{(A 26)}$$

In order to estimate $\|\eta^n\|$ write (A 22) (with n instead of $n+1$) in matrix form:

$$\begin{pmatrix} -r & 1 & 0 & \dots & 0 \\ 1 & -r & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & -r & 1 \\ 0 & \dots & 0 & 1 & -r \end{pmatrix} \begin{pmatrix} \eta_1^n \\ . \\ . \\ . \\ \eta_{J-1}^n \end{pmatrix} = h^2 \begin{pmatrix} \delta_1^n + O(h^2) \\ . \\ . \\ . \\ \delta_{J-1}^n + O(h^2) \end{pmatrix},$$

where $r = 2 + a^2h^2$. With

$$B = \begin{pmatrix} 0 & r^{-1} & 0 & \dots & 0 \\ r^{-1} & 0 & r^{-1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & r^{-1} & 0 & r^{-1} \\ 0 & \dots & 0 & r^{-1} & 0 \end{pmatrix},$$

we obtain

$$\begin{pmatrix} \eta_1^n \\ . \\ . \\ . \\ \eta_{J-1}^n \end{pmatrix} = -(I-B)^{-1}(h^2/r) \begin{pmatrix} \delta_1^n + O(h^2) \\ . \\ . \\ . \\ \delta_{J-1}^n + O(h^2) \end{pmatrix}$$

Now $I-B$ is invertible, since

$$\|B\|_\infty = \max_i \sum_j |b_{ij}| = 2/r < 1.$$

From this we obtain

$$\|(I-B)^{-1}\|_\infty \leqslant \frac{1}{1-\|B\|_\infty} = \frac{r}{a^2h^2}.$$

Hence

$$\|\eta^n\| \leqslant \frac{r}{a^2h^2}\frac{h^2}{r}(\|\delta^n\| + O(h^2)) = \frac{1}{a^2}\|\delta^n\| + O(h^2). \qquad \text{(A 27)}$$

Set $\Delta_n = \max(\|\epsilon^n\|, \|\delta^n\|)$. From (A 25), (A 26) and (A 27) it follows that

$$\Delta_{n+1} \leqslant (1+Ck)\Delta_n + Kk^2$$

for certain constants C and K. Since $\Delta_0 = 0$, iteration of this formula gives

$$\Delta_n \leqslant k(K/C)[(1+Ck)^n - 1] \leqslant k(K/C)(e^{Ct_0} - 1).$$

For the point $(z, t) = (j_\nu h_\nu, n_\nu k_\nu)$ described above we consequently have

$$|\tilde{T}_{j_\nu}^{n_\nu} - T_{j_\nu}^{n_\nu}| = O(h_\nu^2) \quad \text{as} \quad \nu \to \infty,$$

and similarly for w and ϕ. This proves the stated convergence.

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